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ON THE PERFORMANCE OF EXPLICIT AND IMPLICIT ALGORITHMS FOR TRANSIENT THERMAL ANALYSIS OF STRUCTURES

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SUMMARY

The status of an effort to increase the efficiency of calculating transient temperature fields in complex aerospace vehicle structures is described. The advantages and disadvantages of explicit and implicit algorithms are discussed. Explicit solution techniques require minimal computation per time step but have stability-limited step sizes. Implicit techniques permit larger step sizes because of better stability but require more computation per time step. A promising set of implicit algorithms, known as the GEAR package is described. Four test problems, used for evaluating and comparing various algorithms, have been selected and finite element models of the configurations are described. These problems include a Space Shuttle frame component, an insulated cylinder, a metallic panel for a thermal protection system and a model of the Space Shuttle Orbiter wing. Calculations were carried out using the SPAR finite element program, the MITAS lumped parameter program and a special purpose finite element program incorporating the GEAR algorithms.

Results generally indicate a preference for implicit over explicit algorithms for solution of transient structural heat transfer problems when the governing equations are "stiff". Stiff equations are typical of many practical problems such as insulated metal structures and are characterized by widely differing time constants in the thermal response. In cases where implicit algorithms are appropriate, the GEAR algorithms offer high potential for providing increased computational efficiency. In some cases careful attention to modeling detail such as avoiding thin or short high-conducting elements can reduce the stiffness to the extent that explicit methods become advantageous.

INTRODUCTION

An effort is in progress at the NASA Langley Research Center to improve capability to predict and optimize the thermal-structural behavior of aerospace vehicle structures. The focus of this activity is on space transportation vehicles presently typified by the Space Shuttle Orbiter. A principal task is to significantly reduce the computing effort for obtaining transient temperature fields in the structure. This task is to be accomplished by incorporating the best state-of-the-art solution algorithms into general-purpose thermal analysis computer programs. Current activity is focused on evaluation and comparison of explicit and implicit solution algorithms.

In reviewing current literature, a preference is evident among numerical analysis researchers for implicit algorithms for solution of stiff* sets of ordinary differential equations (ODE's). Many engineering analysts, however, prefer to use the longer-established explicit

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*Stiff sets of ordinary differential equations are characterized by solutions with widely varying time-constants. The typical case is when the solution to the homogeneous problem has very small time constants compared to those of the forcing function (ref. 1).

algorithms. A partial explanation for this dichotomy is that the full power of the implicit approach has not been transferred from researchers to engineering analysts.

In the explicit algorithms the set of temperatures at a given time is expressed as an explicit function of the set of previous temperatures in the structure. The time step (the difference between the present and previous times) is limited (often severely) in order that the technique be stable. In the implicit algorithms the present temperatures in the structure are interrelated through a set of algebraic equations (usually nonlinear) which are often costly to solve. For the commonly-used implicit algorithms there is no stability-imposed limitation on step size. The step size is limited by solution accuracy only, so that implicit algorithms can, in general, use much larger time steps than explicit algorithms. Because a single explicit time step is computationally faster than a single implicit time step the key to the advantageous use of implicit algorithms is to use the largest possible time step size.

As presently implemented in thermal analysis computer programs, implicit algorithms generally require a user-specified fixed time step (refs. 2 to 6). The step size must be determined by trial, insight or other means. Because the user is usually unable to choose the largest possible time step at each time point the implicit algorithm is not used to maximum advantage. Furthermore, the solution must be repeated with a smaller time step in order to assess the error in the solution. The lack of automatic selection of step size based on a prescribed error tolerance has certainly delayed the full development of the potential of implicit solution algorithms.

The strategy being advocated in the solution of large problems by implicit methods is to have several alternate implicit algorithms of varying order available and to automatically select both the largest possible time step and the appropriate algorithm throughout the solution process (refs. 6,7). A promising set of algorithms, developed for the purpose of implementing the aforementioned strategy, is denoted the GEAR algorithms (refs. 7 to 10). Good performance of the GEAR algorithms has been demonstrated in applications to problems in structural dynamics, atmospheric pollution and hydrodynamics (ref. 7). These successes suggest the application of the GEAR techniques to transient thermal analysis.

The purpose of the present paper is to describe the current status of ongoing evaluations and demonstrations of the use of explicit and implicit algorithms for transient thermal analysis of heated structures using the finite element method. A Shuttle frame test article, an insulated cylinder, a metallic multiwall thermal protection system panel, and a model of the Shuttle Orbiter wing are analyzed using the SPAR thermal analysis computer code (ref. 2). Comparisons between implicit and explicit algorithms are presented. The performance of the GEAR algorithms is evaluated for the cylinder problem. For benchmark checks the cylinder is also analyzed with the MITAS lumped parameter program (ref. 11). It is a characteristic of thermal analysis by finite element and lumped parameter techniques that careful modeling can minimize stiffness in a problem and conversely, improper modeling can increase the stiffness. Since stiffness is one of the key factors in the performance of implicit and explicit algorithms, evaluations of these algorithms cannot be entirely separated from modeling considerations. Consequently the paper includes a limited study of the effects of modeling on the performance of the explicit and implicit algorithms.

LIST OF SYMBOLS

C	capacitance matrix
DT	time step size
e_n	error in numerical solution of the temperature at time t_n
e_t	truncation error of numerical integration method
F	right hand side of equation for transient problem, see Eq. (1)
G	right hand side of simplified transient problem, $\dot{T} = G(T,t) = C^{-1}F$
h_n	time step
J	Jacobian of system of differential equations = $\partial F/\partial T$
K	conductivity matrix
L	length of a rod element
Q	thermal load vector
q	order of a multistep method
R	residual of the system of equations generated by the implicit method
t	time
t_n	n-th time point
T	vector of temperatures
T_i	temperature at node i
T_o	initial temperature at node
α	thermal diffusivity
α	coefficient in Adams-Moulton method, Eq. (20)
β	coefficient in backward difference method, Eq. (19)

Superscripts

i iteration number

A dot represents differentiation with respect to time

NATURE OF ALGORITHMS USED IN TRANSIENT THERMAL ANALYSIS

A transient heat transfer problem when discretized by a finite element, finite difference or similar technique, is governed by the following system of equations

$$C\dot{T} = Q(T,t) - K(T,t)T = F(T,t) \qquad T(0) \text{ given} \qquad (1)$$

where F is generally a nonlinear function. It is usually impractical to obtain an analytical solution to Eq. (1) so that numerical integration methods are used. These methods obtain an approximation to the solution at discrete time points t_1, t_2, t_3, . . . and are denoted time marching schemes because the solution at a given time is obtained in terms of the values at previous times. The simplest numerical integration technique is the Euler method which uses the first two terms in a Taylor series to predict T at time t_{n+1} as

$$\begin{aligned} T(t_{n+1}) &= T(t_n) + h_n \dot{T}(t_n) \\ &= T(t_n) + h_n C^{-1} F(T(t_n),t_n) \end{aligned} \tag{2}$$

where

$$h_n = t_{n+1} - t_n \tag{3}$$

Euler's method is an example of an explicit integration technique, so-named because $T(t_{n+1})$ is given explicitly in terms of known quantities. Another approach to the numerical integration of equation (1) is the backward difference method which is an example of an implicit method. In this approach

$$\begin{aligned} T(t_{n+1}) &= T(t_n) + h_n \dot{T}(t_{n+1}) \\ &= T(t_n) + h_n C^{-1} F(T(t_{n+1}), t_{n+1}) \end{aligned} \tag{4}$$

Equation (4) is a system of implicit equations for $T(t_{n+1})$, which is generally nonlinear and thus difficult to solve. The explicit algorithm is therefore easier to implement* and in general would be the best choice except for its stability properties. The term stability refers to the error propagation from one time step to the next. A method is unstable when an error in the solution at a time point is magnified at subsequent time points.

To illustrate the problem of stability associated with explicit solution methods, it is instructive to examine the following simple example. Figure 1 shows a 2-node finite element where the temperature of node 2 is given as λt. Based on a linear temperature variation and lumped capacitances, the differential equation for the temperature at node 1 is

$$\dot{T}_1 = \frac{2\alpha}{L^2} (T_2 - T_1) \tag{5}$$

where α is the diffusivity of the material and L is the length of the element. The exact solution to Eq. (5) is

$$T_1 = -\lambda L^2/2\alpha + \lambda t + (T_0 + \lambda L^2/2\alpha)e^{-2\alpha t/L^2} \tag{6}$$

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*The advantage of the explicit algorithm depends in part on the form of the capacitance matrix C. Usually, the capacitances are lumped and C is diagonal. In cases where C is not diagonal each step of the explicit method is much more costly.

which is composed of terms (first two terms) that vary slowly with respect to other terms (last).

To study the stability of the explicit Euler method for this problem, assume an error e_n in $T_1(t_n)$ and calculate the error e_{n+1} in $T_1\ (t_{n+1})$ due to that error. From eqs. (2) and (5)

$$T_1(t_{n+1}) = T_1(t_n) + h_n\ \dot{T}_1(t_n)$$

$$= T_1(t_n)\ [1-2h_n\alpha/L^2] + (\frac{2\alpha h_n}{L^2})\ T_2(t_n) \qquad (7)$$

From Eq. (7)

$$e_{n+1} = (1-2h_n\alpha/L^2)e_n \qquad (8)$$

For stability (that is, no error magnification) it is required that

$$|e_{n+1}| \leqslant |e_n| \qquad (9)$$

or

$$h_n \leqslant L^2/\alpha \qquad (10)$$

For short or thin elements having high diffusivity, eq. (10) imposes a severe limit on the time step which can be taken by the explicit algorithm. For example, a 5mm thick aluminum element (α=7x$10^{-5}$$m^2$/s) requires

$$h \leqslant (5 \times 10^{-3})^2/7 \times 10^{-5} = 0.36 \text{ sec}$$

which is a very small time step when used for temperature histories of several hours.

By contrast, the implicit integration method does not have a stability-limited time step. If the backward difference method,(eq. (4)), is used for eq. (5) one obtains

$$\begin{aligned} T_1(t_{n+1}) &= T_1(t_n) + h_n\ \dot{T}_1\ (t_{n+1}) \\ &= T_1(t_n) + h_n\ (2\alpha/L^2)(T_2(t_{n+1}) - T_1(t_{n+1})) \end{aligned} \qquad (11)$$

From Eq. (11),

$$T_1(t_{n+1}) = [T_1(t_n) + h_n\ (2\alpha/L^2)T_2(t_{n+1})]/(1 + 2h_n\alpha/L^2) \qquad (12)$$

so that if the error in $T_1(t_n)$ is e_n, the error in $T_1(t_{n+1})$ due to e_n is

$$e_{n+1} = e_n/(1 + 2h_n\alpha/L^2) \qquad (13)$$

From Eq. (13) it is clear that for any value of h_n, e_{n+1} will be smaller than e_n.

Another source of error, denoted the truncation error e_t, is due to using only the first two terms in the Taylor series for estimating $T(t_{n+1})$. It is easily shown that this part of the error for both the Euler method and the backward difference method is

$$e_t = \pm 1/2\ h_n^2\ T(t_n) \tag{14}$$

where the minus sign applies for Euler's method and the plus for the backward difference method. Since the exact solution to the example problem is known, the truncation error may be calculated exactly. Eqs. (6) and (14) lead to

$$e_t = \pm(2\alpha^2 h_n^2/L^4)(T_0 + \lambda L^2/2\alpha)e^{-2\alpha t/L^2} \tag{15}$$

For small values of t the exponential is close to unity so that the following condition must be satisfied to avoid large errors.

$$2\alpha^2 h_n^2/L^4 \ll 1 \tag{16}$$

or

$$h_n \ll L^2/\alpha \tag{17}$$

For large values of t, the exponential becomes very small and h can be large without causing a large e_t. In terms of Figure 1, small steps are required early in the temperature history but not later in the history. These conditions immediately suggest the usefulness of variable time steps which are automatically selected according to the local behavior of the temperature response.

This example problem exhibits the essential features of most transient conduction heat transfer problems with respect to the integration techniques, namely:

(1) The thermal response may be divided into regions of slowly and rapidly varying temperatures. Steep transients accompany initial conditions or sudden changes in the heat load.

(2) The rapidity of variation of the transient portion of the temperature history is proportional to the quantity L^2/α. During such a transient, time steps much smaller than L^2/α must be taken no matter what type of integration technique is used.

(3) During a period of slowly-varying temperatures, large time steps may be taken by implicit integration techniques but explicit techniques must still use time steps which are less than L^2/α.

In mathematical terms, the sample problem is an example of a "stiff" problem. A stiff problem is one whose solution includes a slowly varying function of time plus a transient function which changes rapidly. When explicit methods are applied to stiff problems, very small integration time steps must be taken even though the solution changes very slowly. For this reason stiff problems are usually best solved by implicit methods. The effort involved in solving a system such as Eq. (4) is usually cost-effective if a small number of large time steps are used.

The Euler method and the backward difference methods are presented as representatives of a large class of explicit and implicit techniques, respectively. Higher-order methods typically use more previous information to predict the temperature at the current time and have truncation errors which are proportional to higher powers of h_n. Such techniques are called multistep methods and their order is one less than the power of h_n in the truncation error expression. The stability properties of multistep methods are similar to those of the Euler and backward difference methods. Most explicit methods are unstable for time steps much larger than L^2/α. Accordingly, thermal analysis computer programs generally select the explicit time step automatically based on the stability requirement. For implicit methods, accuracy primarily determines the step size, although stability may be a factor for highly nonlinear problems. Even for linear problems some implicit algorithms produce bounded oscillations if the time step size is too large (ref. 14). Often, low-order implicit algorithms are less susceptible to these oscillations. It is concluded that a good package for integrating stiff systems of ordinary differential equations would be one which uses implicit methods and automatically selects the order and the step size based on desired accuracy. One package denoted the GEAR algorithms has these features and is discussed next.

THE GEAR ALGORITHMS

Several software packages based on the work of Gear have been developed for general use (ref. 7). The package most appropriate for application to finite element thermal analysis is denoted GEARIB*. This package is intended to solve systems of ordinary differential equations of the form

$$C(T,t)\,\dot{T} = F(T,t) \tag{18}$$

The package employs two classes of implicit multistep methods, Adams-Moulton and backward differences. For nonstiff equations the Adams-Moulton method of order one through twelve is used. This method has the general form

$$T(t_{n+1}) = T(t_n) + h_n \sum_{i=0}^{q} \beta_i \,\dot{T}(t_{n+1-i}) \tag{19}$$

*An earlier and closely related software package denoted GEARB was developed to solve equations of the form $\dot{T} = G(T,t)$. In the present application $G=C^{-1}F$. At this writing GEARIB has not been implemented and calculations have been performed using GEARB.

where q is the order. For stiff equations the backward difference algorithms of orders one through five are used. These algorithms have the general form

$$T(t_{n+1}) = h_n\beta_0 \dot{T}(t_{n+1}) + \sum_{i=1}^{q} \alpha_i T(t_{n+1-i}) \tag{20}$$

The coefficients α_i and β_i are given in reference 10. The user selects the class of methods (Adams-Moulton or backward differences), and as described in reference 7 GEARIB automatically selects the appropriate time step and the order based on user specified error tolerance. It may seem surprising that implicit methods are used for both stiff and nonstiff problems. However, for nonstiff problems the set of algebraic equations associated with each time step may be solved very effectively and implicit methods often have smaller truncation errors than explicit methods of the same order.

Use of the GEAR algorithms is explained by the backward difference algorithm (of order one). Applied to Eq. (18), eq (20) gives

$$R = C[T(t_{n+1}) - T(t_n)] - h_n F(T(t_{n+1}), t_{n+1}) = 0 \tag{21}$$

This system of nonlinear algebraic equations is solved by the modified Newton's method. That is

$$T^{i+1}(t_{n+1}) = T^i(t_{n+1}) - [\frac{\partial R}{\partial T}]^{-1} R \tag{22}$$

where

$$\frac{\partial R}{\partial T} = C - h_n J$$

$J = \partial F/\partial T$ is the Jacobian of the system at a previous time point and may be calculated according to one of four options specified by the user:

<u>Option 0</u>: The Jacobian is assumed to be the unit matrix. In this case the iteration represented by eq. (22) is very efficient to implement. However, it can be shown that it converges only for very small values of h_n. The upper limit on h_n is of the same order as that required for stability of an explicit method. As a result option 0 is similar in cost and required step size to an explicit method.

<u>Option 1</u>: The Jacobian is calculated in a subroutine by the user. This is the recommended option for stiff problems.

<u>Option 2</u>: The same as option 1 except that the Jacobian is calculated by finite differences. This option is intended for users who do not wish to supply a subroutine to calculate the Jacobian.

Option 3: The Jacobian is assumed to be diagonal and is calculated by finite differences in GEAR. This option is more costly per iteration than option 0 and more efficient than option 1 because $\partial R/\partial T$ is diagonal. Its convergence properties are also in between options 0 and 1.

For options 1, 2 and 3 the Jacobian is recalculated whenever the iterative solution of eq. (21) requires more than three iterations. The Adams-Moulton method has better accuracy but poorer stability than the backward difference method. Therefore for the stiffest problems backward differences with option 1 is generally recommended and for nonstiff problems the Adams-Moulton method with option 0 is recommended(ref. 7). In this paper all applications of the GEAR algorithms use the backward difference method (eq. 20).

DESCRIPTION OF TEST PROBLEMS AND RESULTS

Insulated Shuttle Test Frame

The first test problem used for algorithm evaluation is a Shuttle Orbiter frame analyzed and tested under transient heating as described in reference 12. The configuration shown in figure 2 consists of an aluminum frame surrounded by insulation. The principal purpose of the study of the configuration as discussed in reference 12, was to evaluate the thermal performance of the insulation during a simulated Shuttle flight. A secondary purpose was to evaluate the adequacy of thermal analysis procedures by analytical and test comparisons.

The lumped parameter model received from the author of reference 12 consists of a two dimensional section of a symmetric half of the structure and contains 118 nodes (see figure 2b). The unknown temperatures are located at the centroids of the lumps. The lumped parameter model was converted to a finite element model for analysis using the SPAR program (ref. 2). The corresponding SPAR finite element model contains 149 grid points located at the ends or corners of the elements. The model contains 148 elements including one-dimensional elements which account for conduction in the aluminum structure and radiation across the air gap and two-dimensional elements which model conduction in the insulation and across the gap. The difference in numbers of elements and grid points is due to the different modeling approaches of the two methods.

Minor modifications were made to the finite element model following the conversion. These consisted of eliminating or consolidating some extremely thin or short finite elements in the aluminum structure in order to reduce the stiffness of the equations and to increase the allowable time step for the explicit solution algorithm. The properties of the aluminum structure are functions of temperature and the properties of the insulation are functions of temperature and pressure. Material properties are updated every 50 seconds. The pressure-dependence is treated in SPAR as time dependence since the pressure-vs-time variation is known from the trajectory data for the simulated flight conditions. The applied heating is specified by tabulations of temperatures at the outer surface of the insulation.

The temperature history for the frame was computed using the SPAR explicit (Euler), and implicit techniques - (Crank-Nicholson and backward differences). Comparisons of solution times are given in Table 1. The explicit procedure using a time step of 0.16 s required 513 s of CPU* time. This time step was controlled by conduction through most of the aluminum elements along the center and front of the frame.

Solution time using the Crank-Nicholson algorithm varied from 380 s to 38 s as the time step was varied between 1.0 and 50 sec. The solution times for backward differences were close to those of Crank-Nicholson. As indicated in Table 1(b), there is very little loss of accuracy in either the structure or insulation temperatures with increased time step size. The conclusion is that there is over an order of magnitude difference in solution time between explicit and implicit solution techniques for the frame problem as modelled.

The accuracy of the solutions by the various techniques is further assessed. Figure 3 contains temperature histories at a point in the outer layer of the aluminum structure corresponding to node 309 (see fig. 2b). The solid line in figure 3 represents the applied temperatures at the outer surface of the insulation (node 29). The dotted line shows temperatures obtained by the SPAR analysis. The SPAR temperatures are plotted as a single curve since there is little difference between the results. The dashed-dot line shows analytical results from the lumped parameter analysis of reference 12 which are also in close agreement with the SPAR temperatures. The circular symbols represent test data from reference 12. The closeness of all the results indicates that the models are adequate to simulate the temperature history in the test article.

Insulated Cylinder

Model Description.- For the next test problem, a configuration was sought which was larger (in terms of number of unknown temperatures) than the Shuttle frame and exhibited some of the characteristics of an insulated airframe structure. Also, a simple structure was sought so that a finite element model could be easily generated in a stand-alone program in which the GEARB algorithms could be incorporated and tested. These considerations led to the insulated aluminum cylindrical shell depicted in figure 4. The cylinder is 18m (720 in.) in length and 4.5m (180 in.) in diameter. The aluminum is 0.25cm (0.1 in.) thick and the insulation is 5.0 cm (2.0 in) thick. The outer surface of the insulation is heated over a region which consists of one-third the length and half the circumference. The finite element model consists of a symmetric half of the cylinder and is composed of simple solid elements (K81 elements in SPAR). There are 39 elements along the cylinder length, 4 in the

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*All times are given for the Langley Research Center CYBER 173 computer circumferential direction and 3 through the depth (2 elements in the insulation and one in the structure). Additionally, the outer surface of the insulation has quadrilateral elements (K41) which receive the heat load and quadrilateral radiation elements (R41) which radiate to space. As a result the model contains 800 grid points (hence unknown temperatures) and 650 elements. It is recognized that some features of the model are non optimum. For example modeling the thin aluminum layer with K81 elements and using large high-aspect-ratio elements are not considered good modeling practices. The effects of changing the model to assess these and other shortcomings are discussed later in the paper. The time-dependence of the heat load on the cylinder is plotted in figure 5. For all calculations material properties of the metal and insulation are temperature-dependent and are given in table 2. Material properties are updated every 200 seconds of the temperature history.

Application of GEARB. - The GEARB algorithms were applied to this example using a special purpose finite element program which generates a finite element model of a cylinder using K81, K41, and R41 elements. The program contains the GEARB package and generates the matrix J and the vector G (see footnote, p. 7). Only the backward difference option is used because it is recommended for stiff sets of equations. The first set of calculations concerns selecting the best Jacobian option. Temperature histories in the cylinder were calculated for 2000 s using each of the four options with a specified relative error limit of 0.001. Solution times, integration step sizes, and the number of Jacobian evaluations are given in table 3(a). As expected for this stiff problem, the only useful options are the user-supplied Jacobian (Option 1) and the finite difference Jacobian (Option 2). The degree of stiffness is indicated by the small step size (0.045 s) required by the explicit-like option (Option 0). In contrast, options 1 and 2 permit time steps of up to 93.4 s and average 25 s.

To further investigate the Jacobian options, the problem was made less stiff by increasing the metal thickness to 2.54cm (1.0 in) and the calculations were repeated. The results in Table 3(b) show that options 0 and 3 are acceptable (in fact option 3 is better than option 2 for this case) but are not as effective as option 1. The results indicated in table 3 suggest that as less stiff problems are considered, options 0 and 3 will become more effective. This is an important trend because use of options 0 and 3 requires less core storage than options 1 and 2 since a diagonal Jacobian is used in the latter options.

To assess the effect of accuracy requirements on computation time and results, the thin cylinder was reanalyzed using option 1 and a relative accuracy of 0.01. The CPU time was reduced from 450 s to 263 s, the average time step increased to 69 s and the number of Jacobian evaluations reduced to 9. The largest difference in calculated temperatures resulting from the relaxed error tolerance was only 14K (out of 560K) for a point in the insulation.

Application of SPAR.- The temperature history of the cylinder for 2000 s was computed with SPAR using the explicit Euler algorithm as well as the

Crank-Nicholson and backward difference implicit algorithms. Comparisons of solution times for the methods are shown in table 4. There was essentially no difference between solution times for Crank-Nicholson and backward differences and results are presented only for the former method. The explicit algorithm estimated a stability limit of 0.12 s, however, use of this time step gave an unstable solution. A time step of 0.06 s was used successfully and the solution time was 12300 seconds. The time step was controlled by conduction through the thin aluminum structural elements. The value of L^2/α based on a value of L of 0.254 cm (0.1 in.) was 0.103 s. This example illustrates a case of a stiff problem made more stiff by a model which causes the explicit algorithm to use a small time step to compute the negligible temperature gradient through the aluminum skin. The implicit algorithm was used with time steps of 5, 10 and 25 s and required solution times of 782, 569 and 507 s respectively. For a time step of 50 s the implicit method failed to converge. The relatively small decrease in solution time between time steps of 10 s and 25 s is noted and is due to two reasons. First, a major portion of the time is used in SPAR to regenerate the finite element matrices when material properties are temperature-dependent (see next section). Second, a larger time step often increases the number of iterations required to solve the implicit system (equation 21).

Comparison between GEARB and Implicit Algorithms in SPAR. - Experience with the GEARB algorithms and those presently in SPAR plus comparisons of solution times such as those in Table 4 suggests the following advantages of the GEARB methods:

(i) the use of accuracy-controlled time steps frees the user from the need to determine time steps for achieving desired accuracy;

(ii) The use of variable time steps permits much larger average time steps to be used;

(iii) GEARB employs an efficient predictor (the algorithm that supplies the first guess to the solution of eq. 21) and therefore can save time by employing larger time steps.

To gain additional insight into the benefits of variable time steps and order in GEARB, the cylinder was reanalyzed with the heat load of figure 5 replaced by a step function having the same peak value as figure 5. This load results in a rapid and highly nonlinear response during the first part of the temperature history. Temperature histories were computed using GEARB in the special purpose program and Crank-Nicholson in SPAR (with a time step of 25 s). Material properties were updated every 50 seconds. The time step used in GEARB varied between 3.5 s and 308 s with an average time step of 47 s. The order of the algorithms varied between 3 and 1. The smaller time steps and higher orders were used early in the time history. Solution time for GEARB was 368 s. compared to 1014 s for Crank-Nicholson. Additionally there was an error of 10K (out of 750K) in the Crank-Nicholson result at 50 s.

Effect of Modeling on Algorithm Performance. - The thickness of the cylinder used in the calculations is deliberately chosen to be quite small (consistent with the Shuttle frame for example). For this thickness there is no significant temperature gradient through the aluminum and there is no need to use elements (e.g. K81) which account for the gradient. Additionally, it is possible to replace the three-dimensional K81 elements in the insulation with an assemblage of one dimensional conductors through the insulation thickness. Two models that reflect these ideas were generated. The first model (model II) replaced the 3-dimensional aluminum elements with 2 dimensional (K41) elements. The second model (model III) used the two-dimensional aluminum elements and one-dimensional insulation elements. Both models have a finer (3 elements instead of 2) representation through the insulation so as to preserve the total number of grid points at 800. The solution times with these models are given in the third and fourth columns of Table 5. They indicate that the changes in the model which reduce the stiffness enable the explicit algorithms to execute faster than the implicit algorithms. As noted in the table, the implicit algorithms in SPAR for models II and III did not converge for a time step of 25 s and for model I, the solution time was greater for 25 s than for 17 s. These are additional indications that the predictor used in conjunction with the iterative solution of eq.(21) may be deficient.

Another aspect of the effect of modeling is comparison of results from finite element and lumped parameter models. For this purpose, the MITAS lumped parameter computer program (ref. 11) was applied to the analysis of the cylinder. The finite element model I was converted to a lumped parameter model by use of the CINGEN program (ref. 13).* The resulting lumped parameter model contained 625 nodes as compared to 800 grid points in the finite element model. Recall the unknown MITAS temperatures are located only at the centroids of each lump or element. Temperature histories were obtained using MITAS with the explicit (forward differences) and implicit (Crank-Nicholson and backward differences) methods.

MITAS computation times are shown in the last column of table 5. Because none of the SPAR models is equivalent to the MITAS model in terms of the number of unknown temperature or nodal connections, no direct comparison of MITAS and SPAR solution times are appropriate. However, some trends evident in table 5 are noted. The MITAS model is not particularly stiff as evidenced by the large time step used in the explicit solution technique. The modified SPAR models which begin to resemble the MITAS model in certain respects are also less stiff and favor explicit algorithms. Of particular importance is the decrease in solution time of each program due to increased step size. Specifically note in table 5 the large improvement in solution time between a time step of 10 and 25 seconds in MITAS compared to the much smaller decrease in SPAR.

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*CINGEN did not properly account for two-material conductors. These had to be input manually.

This is primarily due to the fact that SPAR regenerates the element conductivity matrices each time the temperature-dependent conductivities are updated. This results in high computation time especially for the solid (K81) elements. An alternative which can be easily implemented for isotropic elements (and is equivalent to what is done in MITAS) is to generate the matrices once and multiply each matrix by the latest updated conductivity. Presently the time used for the matrix regeneration in SPAR tends to mask some of the benefits of using implicit methods. Namely for larger time steps, the matrix regeneration time becomes a predominant portion of the total solution time.

Figure 6 contains temperature histories of a point in the cylinder computed by the implicit and explicit techniques for all three SPAR models, plus GEARB and MITAS. Model II is considered to be best of the models being compared and thus the temperatures represented by the dotted line are thought to be the most accurate. These results are bracketed by results from model I and MITAS (from above) and by model II (from below). There are negligible differences between temperatures from the implicit and explicit solutions for any given model. Also GEARB produces the same results as SPAR for model I. Results from model II and III are different from that of model I because of the extra layer of elements through the insulation. The MITAS temperature history agrees well with that of model I (on which the MITAS model is based) except for some differences beginning at 1400 s.

Multiwall Thermal Protection System Panel

The next example problem is one which grew out of a study of the thermal performance of a titanium multiwall thermal protection system (TPS) panel which is under study for future use on space transportation systems (ref. 15). The configuration as depicted in figure 7(a), consists of alternating layers of flat and dimpled sheets fused at the crests to form a sandwich. The representation of a typical dimpled sheet is shown in figure 7(b). For the purpose of this analysis, it is assumed that the heat load does not vary in directions parallel to the plane of the panel. This assumption in addition to the regular geometry of the structure leads to the modeling simplification wherein only a triangular prismatic section of the panel needs to be modeled (fig 7(a)). The intersection of this prism with a typical dimpled layer is indicated by the shaded triangle in figure 7(b).

The finite element model shown in figure 8 contains 333 grid points located on nine titanium sheets (5 horizontal and 4 inclined). The model contains 288 triangular and quadrilateral metal conduction elements, 264 solid air conduction elements which account for gas conduction between the layers and 544 triangular and quadrilatral radiation elements which account for radiation heat transfer between adjacent horizontal and inclined sheets. Thermal properties of titanium and air are functions of temperature. Radiation exchange (view) factors were computed and supplied to SPAR using the TRASYS II computer program (ref. 16).

The temperature history of the panel in response to an imposed transient temperature at the outer surface of the panel was computed for 2000 s. Results were obtained with SPAR using explicit, Crank-Nicholson and backward difference algorithms. Solution-time comparisons are presented in table 6. The explicit algorithm required a time step of 0.03 s. This time step was dictated by conduction of heat through the short heat paths between the verticies of adjacent triangular layers and indicates that this is an extremely stiff problem. Required solution time for the explicit algorithm was estimated to be 42000 s.*

The Crank-Nicholson solution was carried out using time steps of 1 and 5 s which led to solution times of 1811 and 675 s respectively. Backward differences was used with the same time steps and had solution times of 1772 and 703 s respectively. This example shows most dramatically the potential advantages of using implicit algorithms for thermal analysis of stiff problems. A plot of typical temperature histories for a point midway through the panel and the primary structure are shown in figure 9 along with the applied outer surface temperature. The results were obtained by the implicit algorithm with a time step of 5 s and are identical to results using a time step of 1 s.

Shuttle Wing

The last example problem is a model of the Space Shuttle Orbiter wing. The model shown in figure 10 is based on a coarse (418 grid point) model and augmented by insulation attached to the upper and lower surfaces. The structure is modeled by rod, triangular and quadrilateral elements (K21,K31,K41 in SPAR terminology). The external insulation on each surface is modeled by five layers of solid triangular prismatic (K61) elements. The complete model contains 2508 grid points, 1400 one-and two-dimensional elements in the structure and 2700 solid elements in the insulation. Thermal properties of the aluminum structure are temperature-dependent; thermal properties of the insulation are temperature and time-dependent.

For the purpose of this analysis, the applied heating on the wing is represented by a time-dependent temperature applied to the external surface of the insulation on the under side of the wing. The shape of this curve shown as the solid line in figure 11 is roughly indicative of atmospheric reentry heating. The temperature history of the wing for 4500 seconds was computed using the SPAR explicit algorithm. Temperature-dependent properties were updated every 100 seconds of the temperature history. For this problem the explicit algorithm was able to use a large time step of 100 s. (The 100 s time step was dictated by the need to periodically update temperature-dependent material properties and not by stability requirements.) The time step is due to the coarse modeling of the structure which did not include the thin, high-conducting or radiating elements present in the previous models. Figure 11 shows the

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*To conserve computer resources the solution was terminated after 400 s of the temperature history for which 8400 CPU s were required.

temperature histories of a point on the structure and a point in the insulation 1/5 of the distance through the insulation at a typical cross section through the wing. The solution time for the explicit algorithm was 8600 s. Next the implicit (Crank-Nicholson) solution algorithm was applied to the wing using a step size of 100 s. Over 5000 seconds of CPU time were used without completing the first time step. It was determined that the excessive slowness was due to the poor banding of the matrix equations which are solved as part of the implicit technique (represented by eq. 21). The grid point decomposition sequence was changed in such a way as to greatly reduce the band width. The implicit solution was repeated with the result that three time steps were completed using 1100 seconds of CPU time. The solution was terminated after this point to conserve computer resources. Extrapolating these values gives an estimate of 16500 CPU seconds to complete the 4500-second temperature history of the wing. Thus the implicit algorithm requires about twice the solution time as the explicit algorithm. This wing problem is a case where because of low stiffness the explicit algorithm is the best choice. It also shows that when using implicit methods, the analyst should be aware of the importance of proper banding of the matrices and careful grid point numbering.

Choice of Explicit or Implicit Algorithms

Two main factors determine whether explicit or implicit algorithms are more effective for solving a structural heat transfer problem. These are (1) stiffness of the system and (2) the connectivity of the model. These are now discussed in detail.

Stiffness of the ODE system. - The stiffer the ODE system is, the more likely it is that the implicit algorithms will be more efficient than explicit algorithms. In many cases careful and judicious modeling of the thermal problem can reduce the stiffness of the resulting system. However, the use of implicit algorithms can help the analyst avoid the added effort required for such a judicious and careful modelling.

The stiffness of the system depends primarily on the smallest time constant (L^2/α) of the elements. Radiation and convection effects increase the stiffness of the model because they increase the conductance without affecting the capacitance. The stiffness of the system also depends on the applied heat loads. The system is stiff if these loads change much more slowly than the smallest time constant of the model. If the loads change very rapidly, small time steps are required to follow the response for both explicit and implicit algorithms. The explicit algorithms most likely will be more efficient in this case.

Effect of connectivity of the model. - The disadvantage of an implicit method is associated with the need to solve a (generally nonlinear) system of equations such as eq. (21) at every time step. The use of the modified Newton method converts this problem to one of solving a series of linear systems. In SPAR, the linear system is solved by Gaussian elimination and in MITAS by an iterative method. When the system of equations is poorly banded, Gaussian elimination is an ineffective solution method. Therefore, in SPAR it may be anticipated that the implicit methods become less attractive for problems which are poorly banded due to poor nodal numbering, or the inherent properties of the model. Problems with inter-element radiation, for example, tend to be poorly banded because non-adjacent grid points are coupled by radiation.

The insulated cylinder problem (model I) is used to demonstrate the effect of banding on the implicit algorithms. The problem was originally modeled with 3 elements through the thickness, 4 in the circumferential direction and 39 in the axial direction. This model has 800 nodes with a band width of 51. The cylinder was remodeled with 9 elements through the thickness, 7 in the circumferential direction and 9 in the axial direction. This model also has 800 nodes but the band width is increased to 182. The solution time using the implicit algorithms for a time step of 5 seconds was 2670 seconds as compared to 782 seconds for the narrow band width cylinder (see table 5). This suggests that an implicit scheme may certainly become less efficient if Gaussian elimination is used as the solution strategy and the system is poorly banded.

CONCLUDING REMARKS

This paper discusses the status of an effort to obtain increased efficiency in calculating transient temperature fields in complex aerospace vehicle structures. Explicit solution techniques which require minimal computation per time step and implicit techniques which permit larger time steps because of better stability are reviewed. A promising set of implicit solution algorithms, known as the GEARB and GEARIB packages are described. Four test problems for evaluating the algorithms have been selected and finite element models of each one are described. The problems include a Shuttle frame component, an insulated cylinder, a metallic panel for a thermal protection system and a model of the Space Shuttle Orbiter wing. Calculations were carried out using the SPAR finite element program, a special purpose finite element program incorporating the GEARB algorithms, and for checking purposes the MITAS lumped parameter program.

Results generally indicate that implicit algorithms are more efficient than explicit algorithms for solution of transient structural heat transfer problems when the governing equations are stiff. Stiff equations are typical of many practical problems such as insulated metal structures and are characterized by widely differing time constants and cause explicit methods to take very small time steps. In those cases where implicit algorithms are appropriate, the GEARB and GEARIB algorithms offer high potential for obtaining the increased computational efficiency.

Studies were also made of the effect on algorithm performance of different models of the same cylinder test problem. These studies revealed that the stiffness of the problem is highly sensitive to modeling details and that careful modeling can reduce the stiffness of the resulting equations to the extent that explicit methods are advantageous. Since implicit algorithms are less influenced by stiffness-related modeling details, use of these algorithms can save the analyst a certain amount of model refinement effort. Finally, wide-banding of the matrix equations of the finite element model either due to non-optimal grid-point numbering or high connectivity (due for example to radiation) may decrease the advantage of implicit methods.

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Table 1. - Performance of Various Algorithms for Transient Thermal Analysis of Shuttle Frame

(a) Solution Time Comparison

EXPLICIT		IMPLICIT			
Euler		Crank-Nicholson		Backward Difference	
Time Step (s)	Solution Time (s)	Time Step (s)	Solution Time (s)	Time Step (s)	Solution Time (s)
0.16	513	1	380	1	357
		10	65	10	68
		25	48	25	49
		50	38	50	41

(b) ffect of Time Step on Accuracy of Implicit Algorithms

Step Size (s)	Temp. of Node 309** at 1200 s		Temp. of Node 409** at 1200 s	
	K	°F	K	°F
1.0	436.2	325.2	528.4	491.1
10.0	436.2	325.1	528.4	491.0
25.0	436.1	325.0	528.3	491.0
50.0	437.2	437.0	528.6	491.5
0.16*	437.8	328.0	529.0	492.2

* Explicit Algorithm
** See figure 2(b)

Table 2. - Material Properties for Insulated Cylinder

(a) Insulation: ρ = 160 kg/m^3 (.00582 lbm/in^3)

T		C		k	
K	°R	J/kg-°C	Btu/lbm-°R	w/m-°C	Btu/in-s-°R
456	360	523	0.125	.0381	5.1x10^{-7}
622	660			.0546	7.3
733	860			.0711	9.5
844	1060			.0898	1.2x10^{-6}
956	1260			.112	1.5
1067	1460			.142	1.9
1778	1660			.180	2.4

(b) Aluminum: ρ = 2770 kg/m^3 (.101 lbm/in^3)

456	360	769	0.184	99.5	.00133
567	560	861	.206	125.0	.00167
622	660	903	.216	138.0	.00185
678	760	937	.224	154.8	.00207
733	860	974	.233	171.3	.00229
789	960	1012	.242	178.8	.00239
844	1060	1045	.250	181.1	.00242

Table 3. - Effect on Soluction Time of Various GEARB Options for Jacobian Evaluation for Insulated Aluminum Cylinder

(a) 0.254 cm (0.1 in.) Aluminum thickness

Jacobian Option	Unit Matrix (0)	User Supplied (1)	Finite Difference (2)	Finite Diff (Diag) (3)
CPU Time* (s)	30,000**	450	955	10,000***
Step Size- Range Average	-- 0.045	15-93.4 25.0	15.0-93.4 25.0	-- 0.8
Number of Jacobian Evaluations	0	17	17	2600
(b) 2.54 cm (1.0 in.) Aluminum thickness				
CPU Time* (s)	1075	402	810	703
Step Size- Range Average	1.6-14.8 2.2	13.8-83.2 30.8	13.8-83.2 30.8	2.2-21.9 8.1
Number of Jacobian Evaluations	0	14	14	223

* For CDC CYBER 173 Computer
** Estimate based on 1540 s for 100 s of History
*** Estimate based on 1028 s for 200 s of History

Table 4. - Solution Times for Various Algorithms In Transient Thermal Analysis of Insulated Cylinder

Explicit Euler		Implicit			
		Crank-Nicholson/ Backward Difference		GEARB	
Time Step (s)	Solution Time (s)	Time Step (s)	Solution Time (s)	Time Step (s)	Solution Time (s)
0.06	12300	5 10 25	782 569 507	Variable: 15-93.4 29-172	 450* 263**

Solution Times on Langley CDC CYBER 173 Computer System

* Specified relative error tolerance 0.001
** Specified relative error tolerance 0.01

Table 5. - Effect of Modeling on Solution Times* for Insulated Cylinder Problem

Program and Model / Algorithm	SPAR (Ref. 2)			MITAS (Ref 11)
	Model I- 3-D metal and insulation elements	Model II- 2-D metal elements, 3-D insulation elements	Model III- 2-D metal elements, 1-D insulation elements	lumped parameter model
Explicit (time step)	12,300 (.06)	388 [illegible]4-40.)	77 (4.6-40.)	92 (25)
Implicit** (DT=5s)	782	770	403	387
Implicit** (DT=10s)	569	556	260	238
Implicit** (DT=17s)	482	534	216	166
Implicit** (DT=25s)	507	Non Convergence	Non Convergence	125

* Time in seconds for CDC CYBER 173 Computer

** Crank-Nicholson and Backward Differences

Table 6. - Comparison of Algorithms for Transient Thermal Analysis of Titanium Multi[illegible] TPS

EXPLICIT		CRANK-NICHOLSON		BACKWARD DIFFERENCES	
Time Step (s)	Solution Time (s)	Time Step (s)	Solution Time (s)	Time Step (s)	Solution Time (s)
0.03	42,000*	1	1811	1	1772
		5	675	5	703

* Estimate Based on 8400 CPU s for 400 s of Temperature History

Solution Times for CDC CYBER 173 Computer

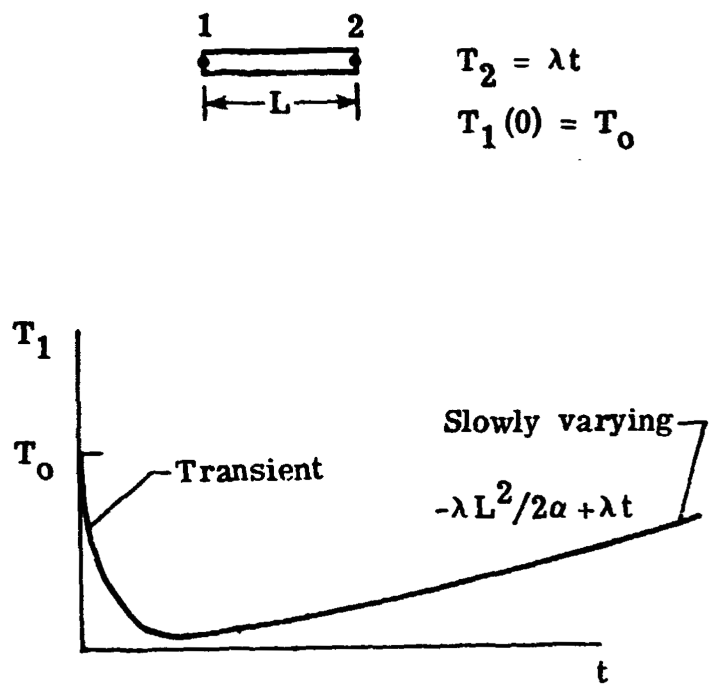
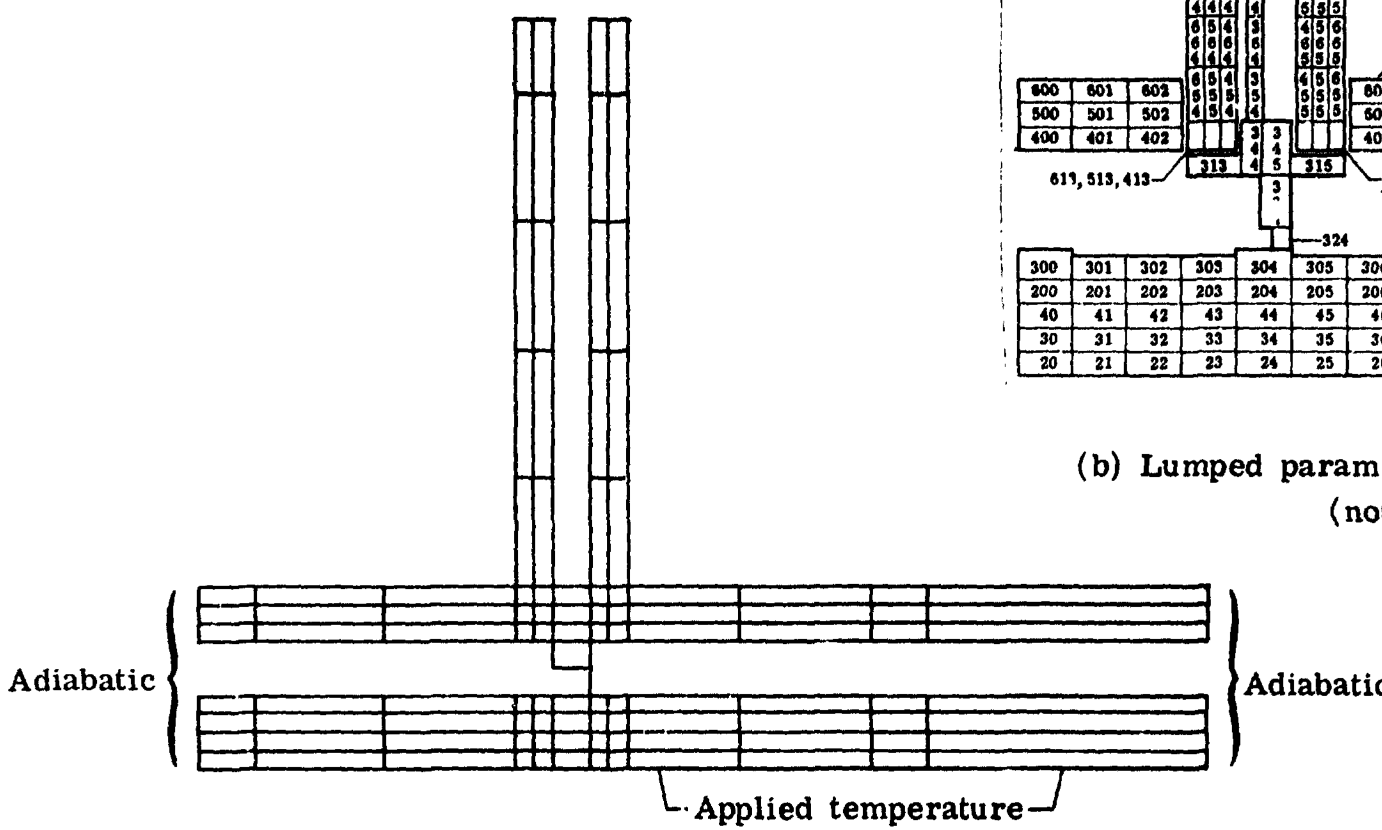
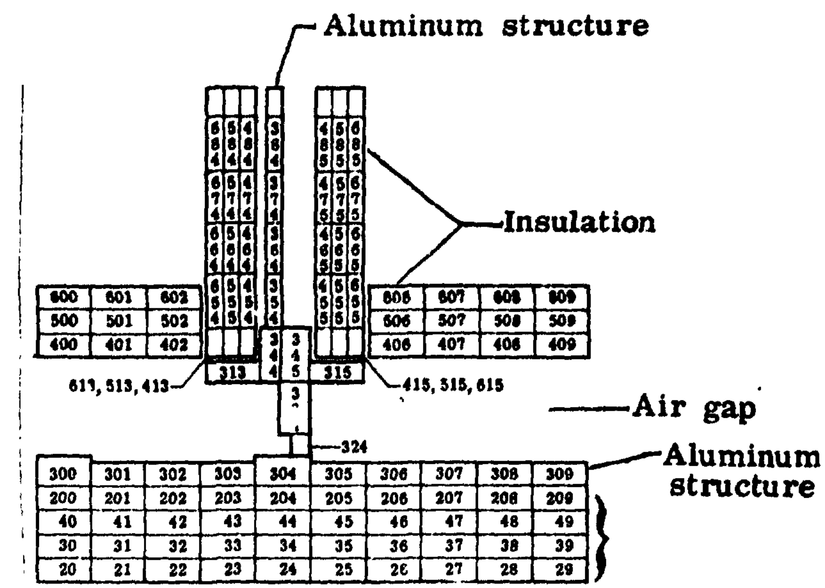


Figure 1.- Temperature history for bar example.



(a) Finite element model.



(b) Lumped parameter model from Ref. 12
(not to scale)

Figure 2.- Finite element and lumped parameter models of shuttle frame.

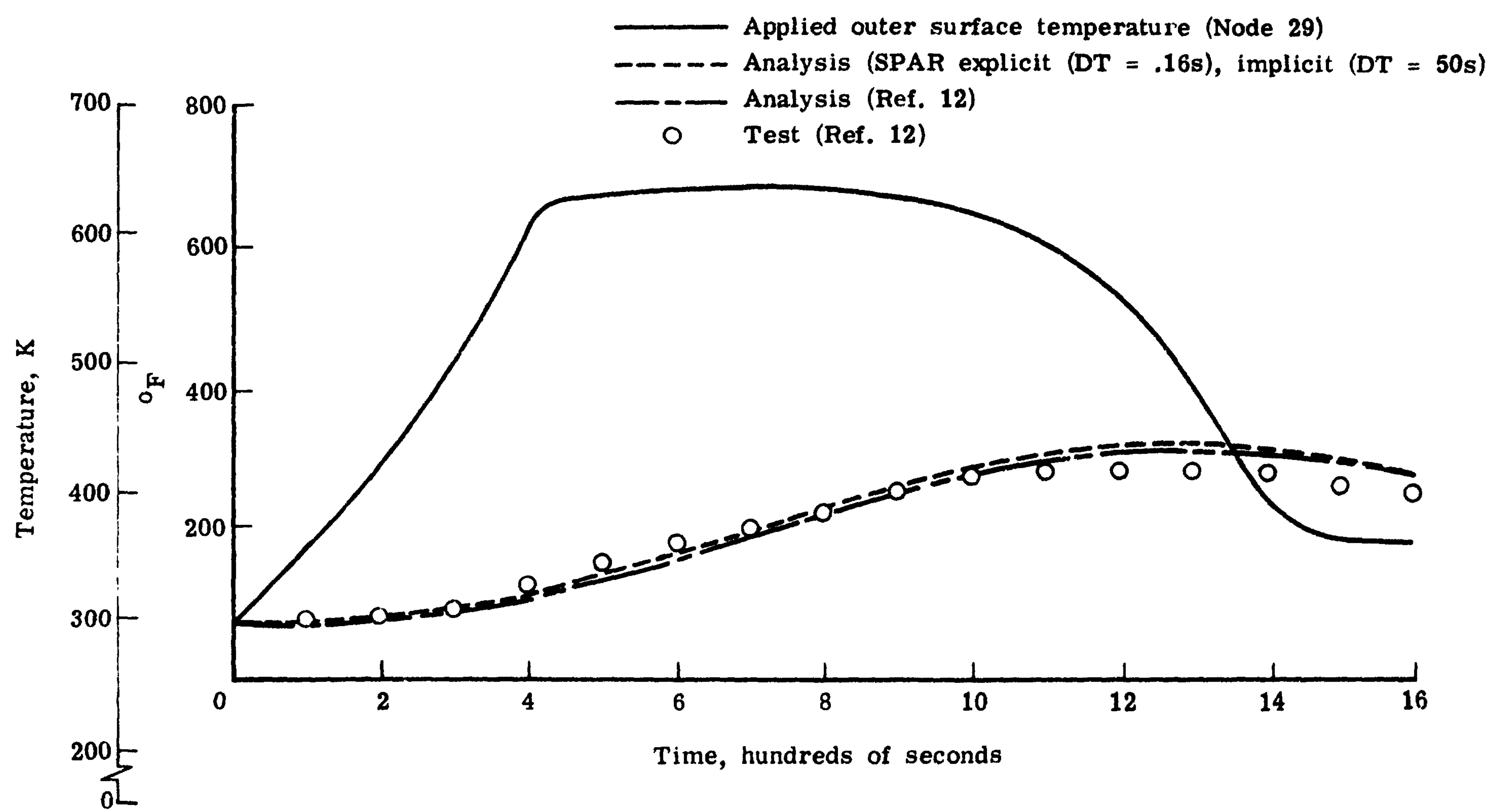


Figure 3.- Temperature history in outer structural surface of shuttle frame (Node 309).

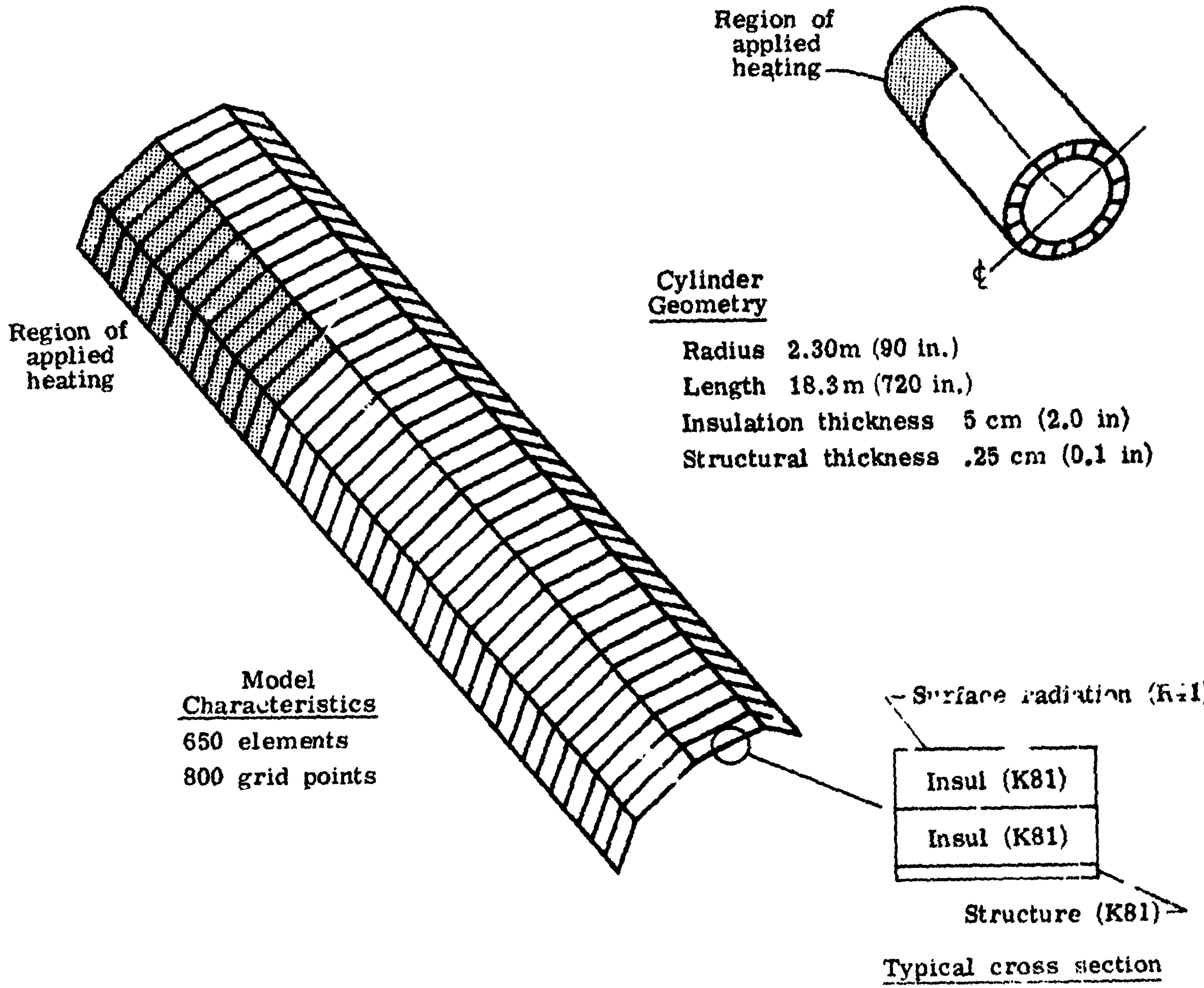


Figure 4.- Finite element model of insulated cylinder.

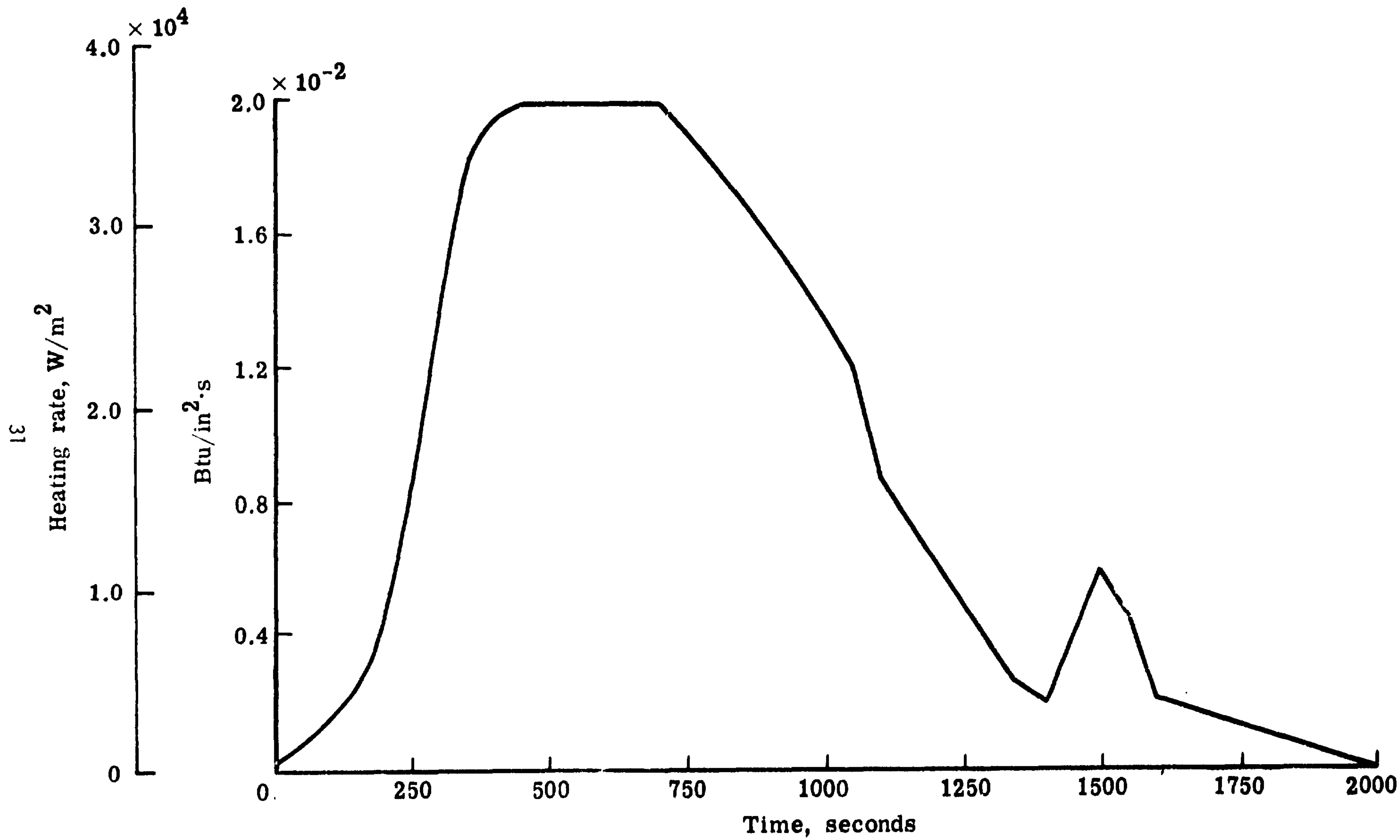


Figure 5.- Heating load at outer surface of insulated cylinder.

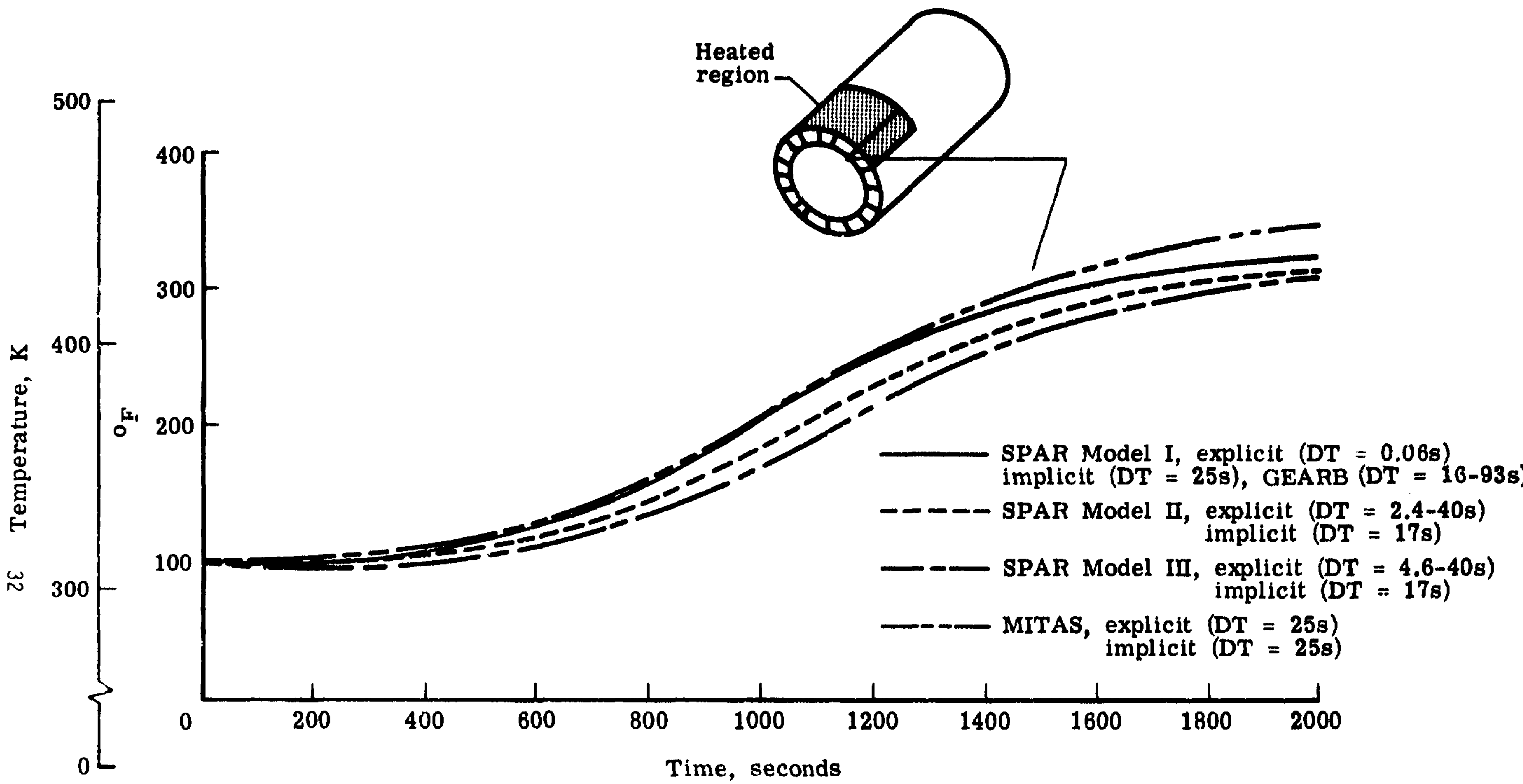
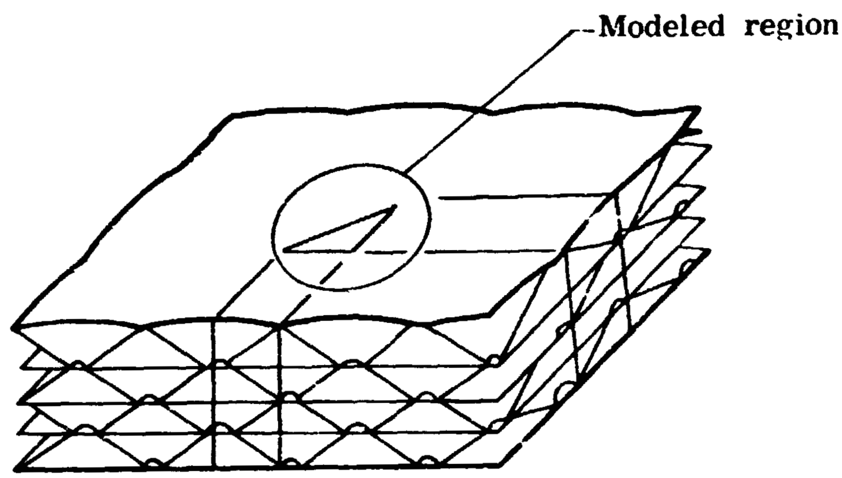
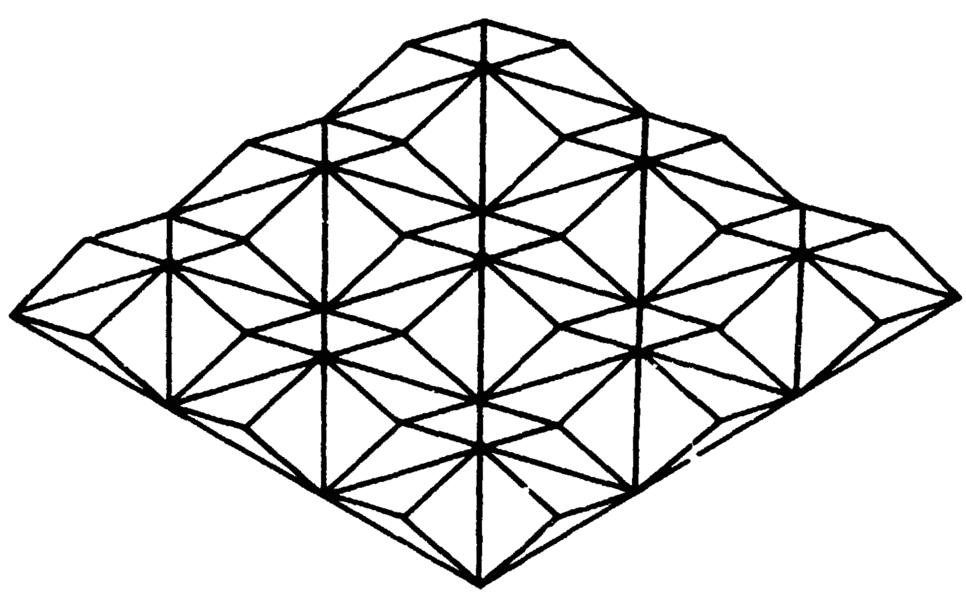


Figure 6.- Effects of choice of algorithm and model changes on temperature history of insulated cylinder. Model I: all 3D elements. Model II: insulation- 3D, metal- 2D. Model III: insulation- 1D, metal- 2D.



(a) Overall construction.



(b) Representation of dimpled layer.

Figure 7.- Titanium multiwall thermal protection system.

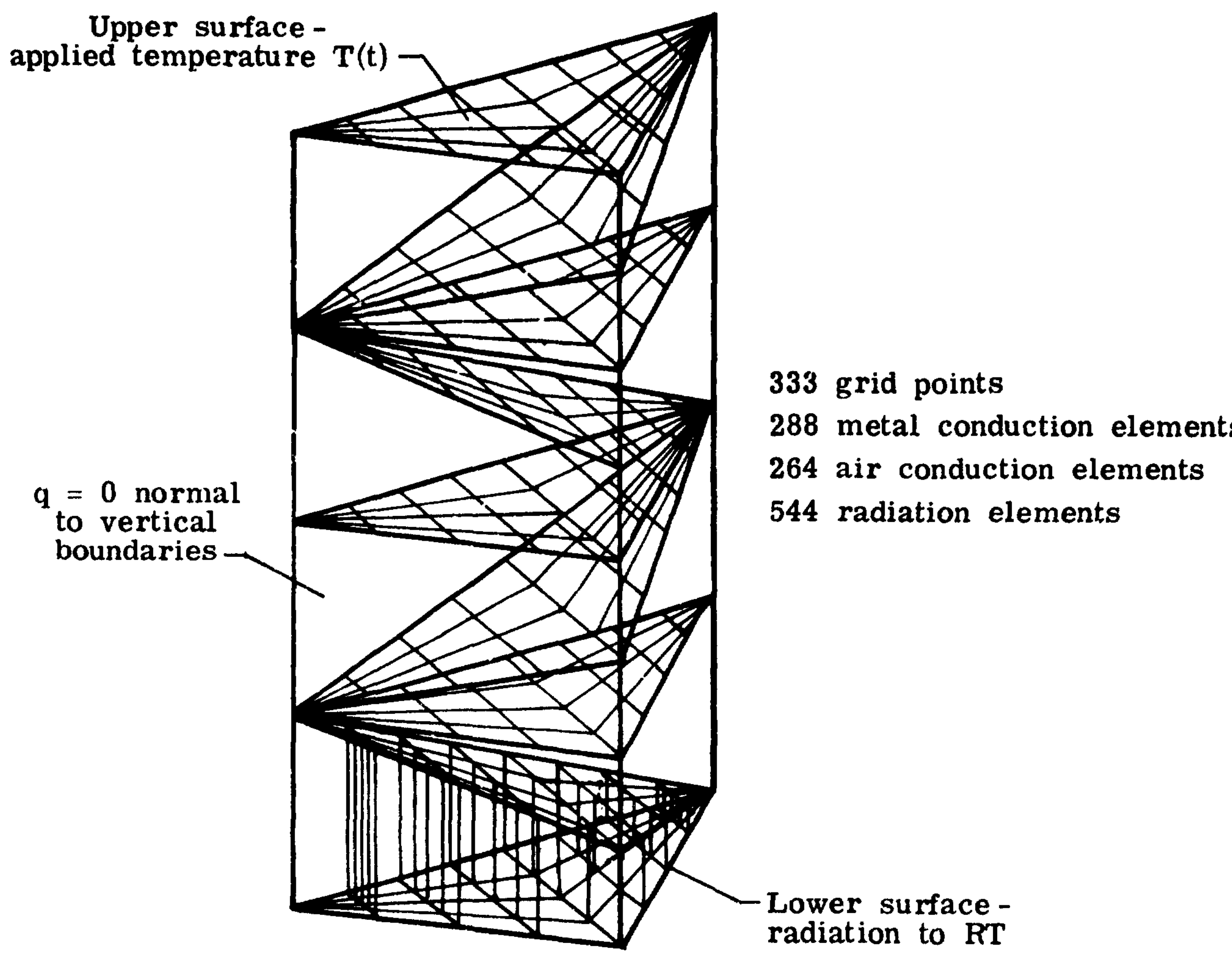


Figure 8.- Finite element model of titanium multiwall TPS.

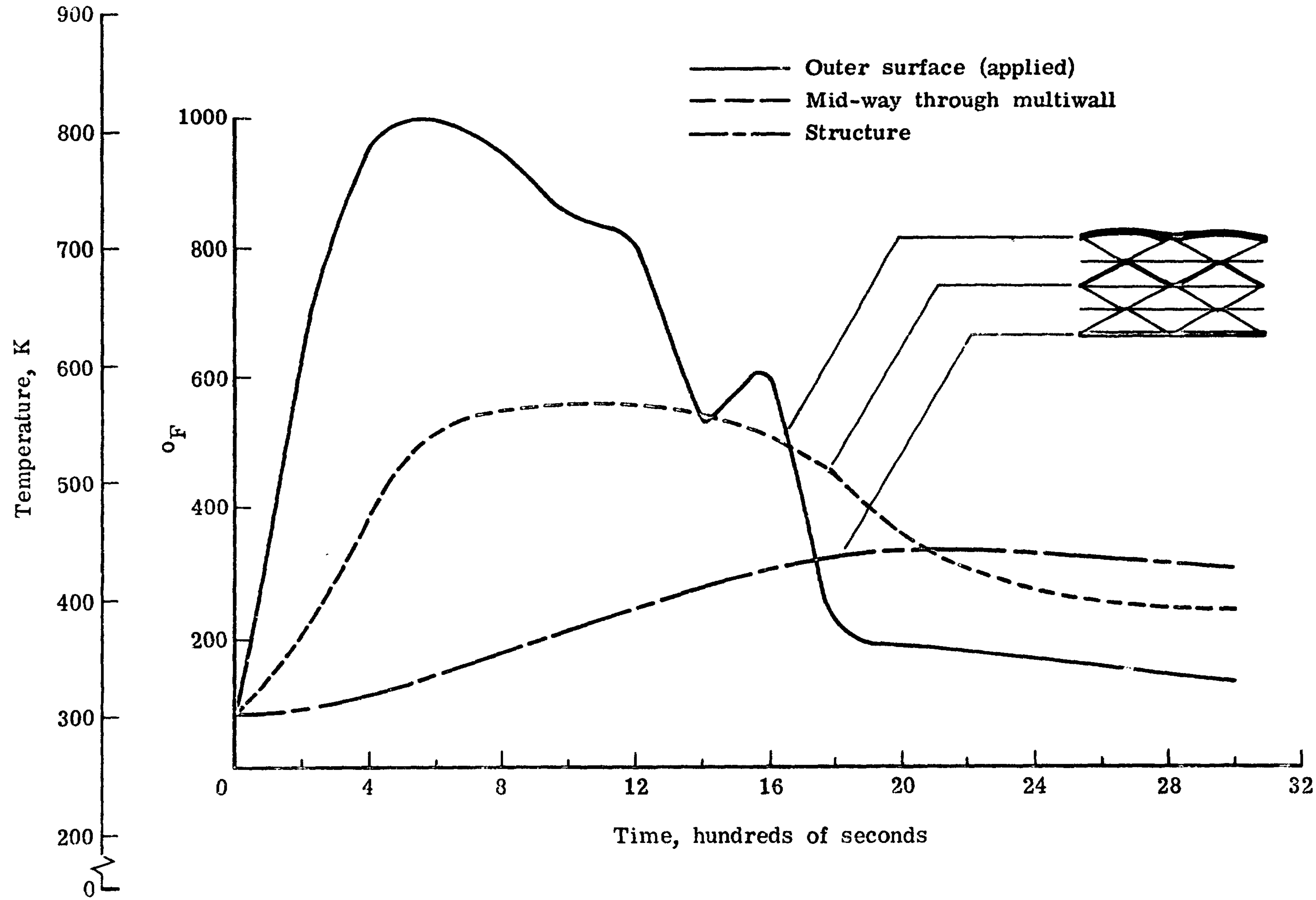


Figure 9.- Transient temperatures in titanium multiwall TPS Panel.

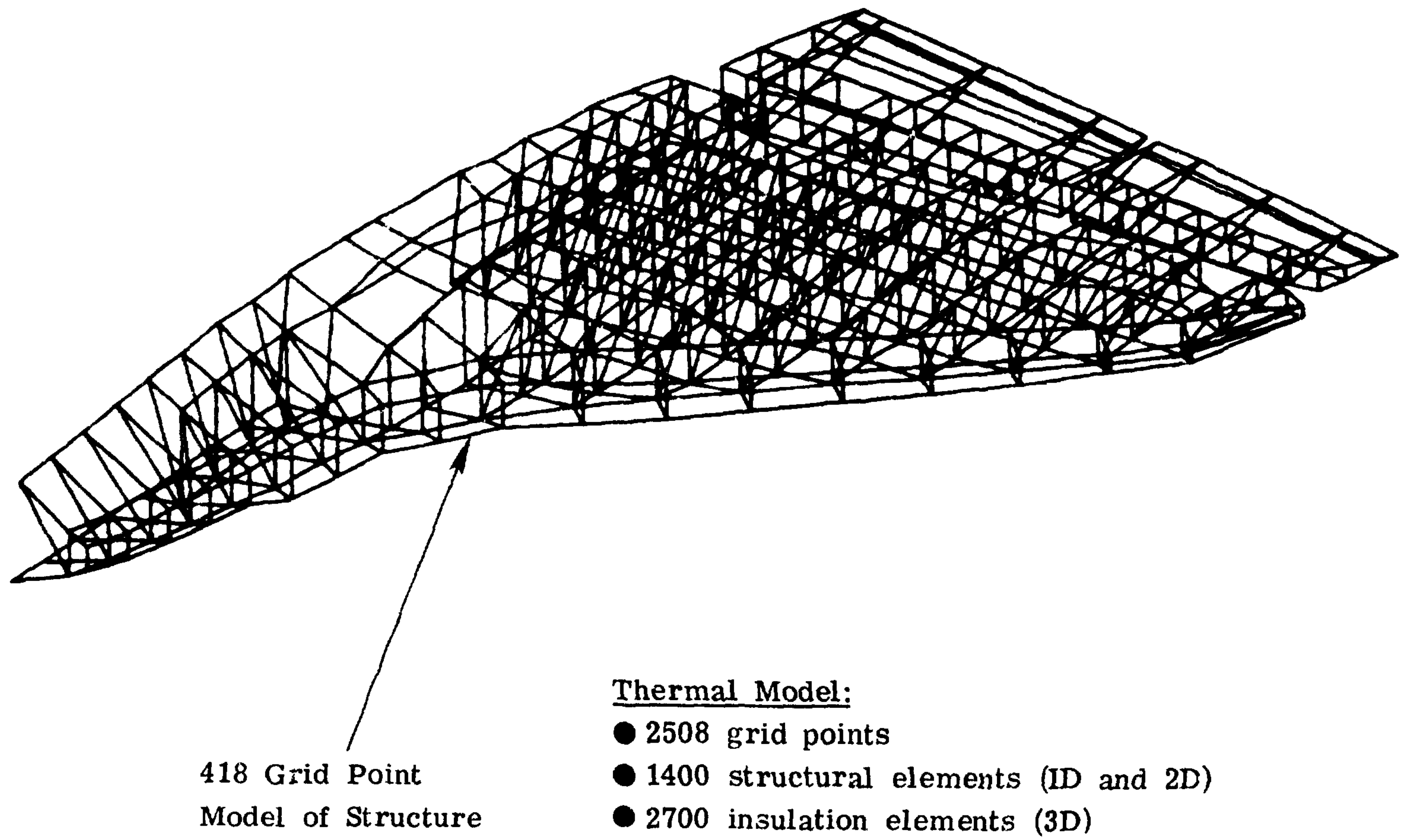


Figure 10.- Finite element model of shuttle orbiter wing.

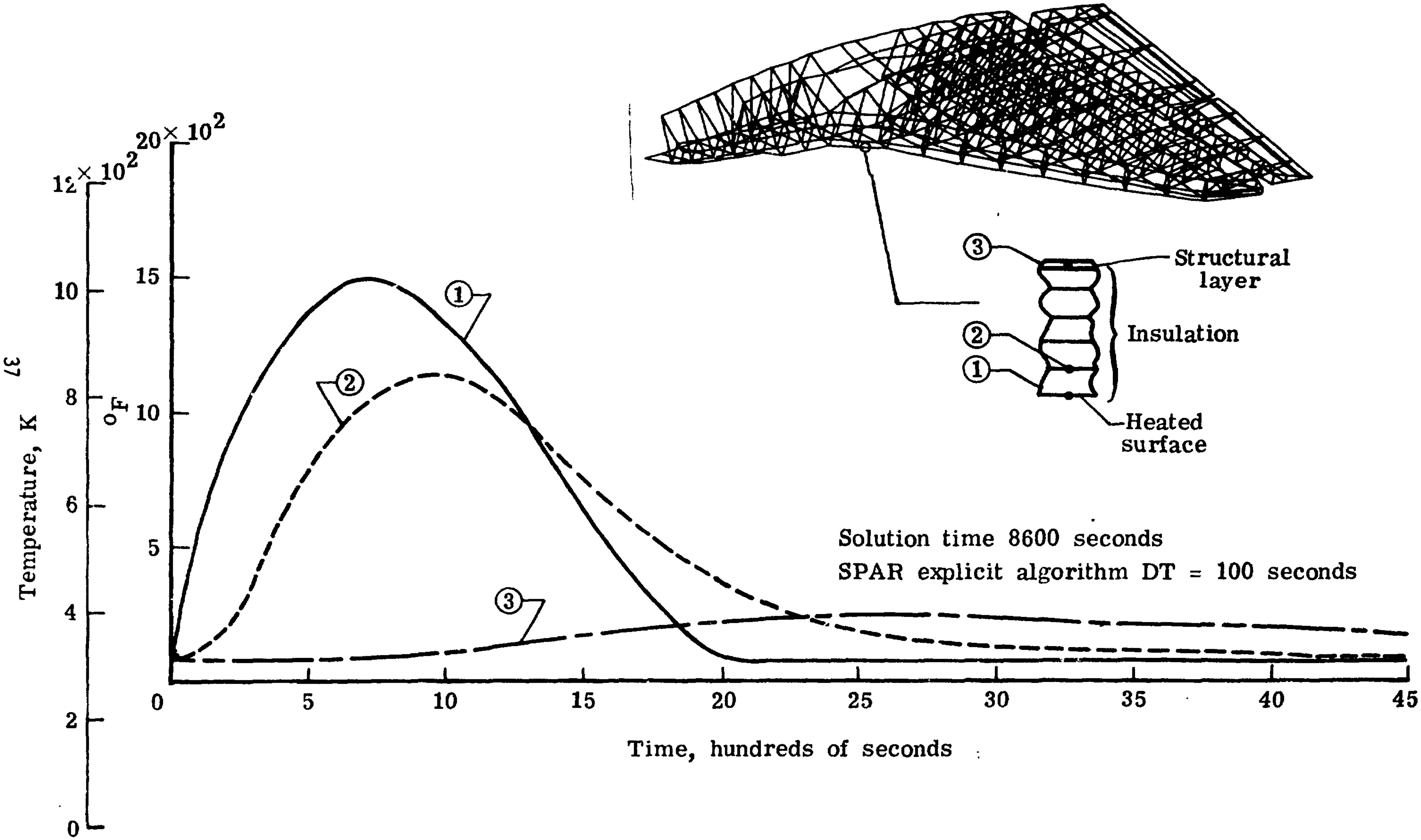


Figure 11.- Transient temperatures in shuttle orbiter wing.

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16. Abstract

The status of an effort to increase the efficiency of calculating transient temperature fields in complex aerospace vehicle structures is described. The advantages and disadvantages of explicit and implicit algorithms are discussed. A promising set of implicit algorithms, known as the GEAR package is described. Four test problems, used for evaluating and comparing various algorithms, have been selected and finite element models of the configurations are described. These problems include a Space Shuttle frame component, an insulated cylinder, a metallic panel for a thermal protection system and a model of the Space Shuttle Orbiter wing. Calculations were carried out using the SPAR finite element program, the MITAS lumped parameter program and a special purpose finite element program incorporating the GEAR algorithms.

Results generally indicate a preference for implicit over explicit algorithms for solution of transient structural heat transfer problems when the governing equations are "stiff." Stiff equations are typical of many practical problems such as an insulated metal structure and are characterized by widely differing time constrants in the thermal response. Careful attention to modeling detail such as avoiding thin or short high-conducting elements can sometimes reduce the stiffness to the extent that explicit methods become advantageous.

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